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Spectral results for some Hausdorff matrices

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Let $BV[0, 1]$ denote the Banach space of functions f of bounded variation over the interval $[0, 1]$, with $\|f\| = V(f)$, the variation of f over $[0, 1]$. In addition, each f is assumed to satisfy $f(t) = [f(t+0) + f(t-0)]/2$ at each point of discontinuity for $0 < t < 1$, and $f(0+) = f(0) = 0$. $\mathfrak{A}$ will denote the subspace of $BV[0, 1]$ consisting of absolutely continuous functions. Let $\mathscr{H}$ denote the Banach algebra of multiplicative conservative Hausdorff matrices. For $f \in BV[0, 1]$, let H_f denote the Hausdorff matrix corresponding to the moment sequence $\{\mu_n\}$, defined by $\mu_n = \int_0^1 t^n \, df$, $n = 0, 1, 2, \ldots$, and define $T_f(z) = \int_0^1 t^{-1+1/z} \, df$ for each $z \in \bar{D} - \{0\}$, where $D = \{z: |z - 1/2| < 1/2\}$.

The Euler matrix of order q, written (E, q), is the Hausdorff matrix corresponding to

$$\varphi_q(t) = \begin{cases} 0, & 0 \leqq t < q, \\ 1, & q \leqq t \leqq 1, \end{cases} \quad 0 < q < 1.$$

The corresponding function for I, the identity matrix is

$$\varphi_1(t) = \begin{cases} 0, & 0 \leqq t < 1, \\ 1, & t = 1. \end{cases}$$

M will denote the Cesaro matrix of order 1, corresponding to $f(t) = t$, $0 \leqq t \leqq 1$.

In [11] it was shown that $\mathscr{H}$ can be identified with $BV[0, 1]$, and that $\mathfrak{A}$ can be identified with $L^1[0, 1]$, the Banach space of absolutely integrable functions on $[0, 1]$. In [5] it was shown that $\|H_f\| = V(f)$. For additional properties of Hausdorff matrices the reader may consult [3].

For $f, g \in BV[0, 1]$, define

$$(f * g)(t) = f(t) g(1) - \int_t^1 g(u) \, df(t/u), \quad 0 \leqq t \leqq 1. \tag{1}$$

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The multiplication defined by (1) is a reformulation, and an extension to multiplicative Hausdorff matrices, of the formula developed in [2, p. 196].

Theorem 1. *Let* $f, g \in BV[0, 1]$. *Then*
(a) $H_{f*q} = H_f \cdot H_g$,
(b) $\|f*g\| \leqq \|f\| \|g\|$, *and*
(c) $f*g = g*f \in BV[0, 1]$.

Since $f, g \in BV[0, 1]$, there exist moment sequences $\{b_n\}$, $\{c_n\}$ defined by $b_n = \int_0^1 t^n \, dg$, $c_n = \int_0^1 t^n \, df$, $n = 0, 1, 2, \ldots$. From Theorem 210 of [3], the sequence $\{a_n\}$, defined by $a_n = b_n c_n$, is also a moment sequence. Therefore, there exists a mass function $h \in BV[0, 1]$ such that $a_n = \int_0^1 t^n \, dh$. To prove (a), it remains to show that $h = f*g$.

If one defines $b(z) = \int_0^1 t^z \, dg(t)$, $c(z) = \int_0^1 t^z \, df(z)$, then $b(z)$ and $c(z)$ are analytic for $\operatorname{Re} z > 0$ and continuous for $\operatorname{Re} z \geqq 0$. Since $a_n = b_n c_n$ for all nonnegative integers n, it follows that $a(z)$, defined by $a(z) = b(z) c(z)$, has the representation $a(z) = \int_0^1 t^z \, dh(t)$.

By defining $u = e^{-t}$, $v = e^{-s}$, $h(u) = h(1) - A(t)$, $g(u) = g(1) - B(t)$, and $f(u) = f(1) - C(t)$, where A, B, and C are as defined on pages 200—201 of [2], the equation

$$\int_0^1 t^z \, dh(t) = \int_0^1 t^z \, dg(t) \cdot \int_0^1 t^z \, df(t)$$

becomes

$$\int_0^\infty e^{-tz} \, dA(t) = \int_0^\infty e^{-tz} \, dB(t) \cdot \int_0^\infty e^{-tz} \, dC(t).$$

Using the multiplication theorem derived on page 201 of [2], it follows that $\int_0^\infty e^{-zt} \, dA(t) = z \int_{-\infty}^\infty e^{-zt} \, dA(t)$, where

$$A(t) = \int_0^t B(t-s) \, dC(s). \tag{2}$$

Substituting the values of A, B, C, u, v into (2), and noting that $B(t-s) = g(1) - g(u/v)$, yields

$$h(1) - h(u) = \int_1^u [g(1) - g(u/v)](-df(v)) = -g(1) \int_1^u df(v) + \int_1^u g(u/v) \, df(v) =$$
$$= g(1) \int_u^1 df(v) - \int_u^1 g(u/v) \, df(v).$$

Therefore

$$h(u) = h(1)-g(1)[f(1)-f(u)]+\int_u^1 g(u/v)\,df(v) =$$

$$= h(1)-g(1)f(1)+f(u)g(1)-\int_u^1 g(t)\,df(u/t),$$

and (1) is established, provided it can be shown that $h(1)=g(1)f(1)$. But this is easy. Since $a_n=b_nc_n$ for all n, $a_0=b_0c_0$; i.e.,

$$\int_0^1 dh(t) = \int_0^1 dg(t)\cdot\int_0^1 df(t),$$

so that $h(1)-h(0)=\big(g(1)-g(0)\big)\big(f(1)-f(0)\big)$. But $h(0)=g(0)=f(0)=0$, so that $h(1)=g(1)f(1)$.

To prove (b), $\|f*g\|=\|H_{f*g}\|=\|H_f\cdot H_g\|\leqq\|H_f\|\cdot\|H_g\|=\|f\|\cdot\|g\|$.

Since multiplication of Hausdorff matrices is commutative, (c) follows immediately from (a) and (b).

Definition (1) is useful, not only for computing mass functions for products of moment sequences, but also is useful as a tool for computing the spectra for particular Hausdorff matrices, as the following theorem illustrates.

Theorem 2. *The spectrum of* M, $\sigma(M)=\{z\colon |z-1/2|\leqq 1/2\}$.

Suppose there exists a mass function $f\in BV[0, 1]$ such that $\big((t-\lambda\varphi_1)*f\big)(t)=$ $=\varphi_1(t)$ for some complex number λ. Then, for $0<t<1$, $tf(1)=t\int_t^1 \big(f(u)/u^2\big)\,du=$ $=\lambda f(t)$. Hence

$$f(1)+\int_t^1 \frac{f(u)}{u^2}\,du = \lambda\frac{f(t)}{t},$$

which implies

$$-\frac{f(t)}{t^2} = \lambda\left[-\frac{f(t)}{t^2}+\frac{f'(t)}{t}\right]$$

a.e. in (0,1). For $\lambda\neq 0$ the above equation takes the form $f'(t)/f(t)=(\lambda-1)/\lambda t$, which has the solution $f(t)=At^{(\lambda-1)/\lambda}$ for some constant A. For Re $\big((\lambda-1)/\lambda\big)>0$, $f\in BV[0, 1]$ and, for $\mathrm{Re}\big((\lambda-1)/\lambda\big)<0$, $f\notin BV[0, 1]$. Since Re $\big((\lambda-1)/\lambda\big)>0$ is equivalent to $\lambda\notin\bar{D}$, and since the spectrum is always closed, the theorem is proved.

Remarks. 1. Theorem 2 is not a new result, but the proof is new. A different proof appears in [4]. Theorem 2 can also be established using the techniques employed in Theorem 4 of [6]. Since M is also a weighted mean method, Theorem 2 is a special case of Theorem 1 of [1].

2. Using the same technique as in Theorem 2, it can be shown that $\sigma(E, q)= =\{z: |z|\leqq 1\}$, a result established by a different method in Theorem 3 of [9]. Also, if $f(t)=t^k$, $k>0$, then $\sigma(H_f)=\bar{D}$.

Theorem 3. *Let* $f\in BV[0, 1]$ *such that* $f(t)/t\in L^1[0, 1]$. *Then* $f(t)*t\in\mathfrak{A}$.

From (1), $f(t)*t=tf(1)+h(t)$, where $h(t)=t\int_t^1 (f(u)/u^2)\,du$. Then, a.e. on $[0, 1]$,

$$h'(t)=\int_t^1 \frac{f(u)}{u^2}\,du-\frac{f(t)}{t},$$

and

$$\int_0^1 |h'(t)|\,dt\leqq \int_0^1\int_t^1 \frac{|f(u)|}{u^2}\,du\,dt+\int_0^1\frac{|f(t)|}{t}\,dt.$$

Interchanging the order of integration in the first integral yields the second integral, so $h'\in L^1[0, 1]$. Thus, $h\in L^1[0, 1]$ and the theorem follows.

It follows from [7] that the set of conservative Hausdorff matrices is a maximal commutative Banach subalgebra of the algebra of conservative matrices. Consequently, the spectrum of any member of $\mathscr{H}$ is determined by the set of multiplicative linear functionals defined in this subalgebra. (See, e.g. [12, p. 264].) The remainder of this paper is devoted to a study of these functionals, and in extending some of the results of [10].

Theorem 4. *Let* χ *be any multiplicative linear functional defined in* $\mathscr{H}$ *such that* $\chi(M)\neq 0$. *Then* $\chi(T_fM)=T_f(\chi(M))$ *for each* $f\in BV[0, 1]$.

Without loss of generality it may be assumed that f is nondecreasing. Define $f_\delta(t)=0$ for $0\leqq t<\delta$, $f_\delta(t)=f(t)$ for $\delta\leqq t\leqq 1$. Then $f_\delta\in BV[0, 1]$ and $\|f-f_\delta\|\leqq \leqq f(\delta)$. Since $f(t)\to 0$ as $t\to 0$, $\|f-f_\delta\|\to 0$ as $\delta\to 0$. Therefore $\|H_f-H_{f_\delta}\|\to 0$ as $\delta\to 0$. Since $|T_f(z)-T_{f_\delta}(z)|\leqq\|H_f-H_{f_\delta}\|$ for each $z\in\bar{D}-\{0\}$, $T_{f_\delta}(z)\to T_f(z)$ as $\delta\to 0$.

Define $\chi(M)=z$, $z\in\bar{D}-\{0\}$. Since clearly $f(t)/t\in L^1[0, 1]$, $f_\delta(t)*t\in\mathfrak{A}$ by Theorem 3. As in the proof of Corollary 4 of [10], $\chi(MH_{f_\delta})=zT_{f_\delta}(z)$. Since $z\neq 0$, $\chi(H_{f_\delta})=T_{f_\delta}(z)$. Taking the limit as $\delta\to 0$ yields $\chi(H_f)=T_f(z)$, and the proof is finished, since $H_f=T_f(M)$.

Remark 3. In Remark 3 of [10] it was shown that, for $z\in D$, $\chi(M)=z$ implies $\chi(T_fM)=T_f(\chi(M))$. The above theorem has extended this result to $\bar{D}-\{0\}$. That Theorem 4 cannot be extended to $\bar{D}$ will be shown in the example following Remark 4.

Define χ_0 by $\chi_0(H_f)=\lim\mu_n$, where $\mu_n=\int_0^1 t^n\,df$. Then χ_0 is a nonzero multiplicative linear functional on $\mathscr{H}$ with the property that $\chi_0(E, q)=0$ for each $0<q<1$.

Theorem 5. *Let χ be any nonzero multiplicative linear functional on $\mathscr{H}$. Then $\chi(E, q)=0$ for some $0<q<1$ if and only if $\chi=\chi_0$.*

If $\chi=\chi_0$ then clearly $\chi(E, q)=0$. Suppose $\chi(E, q)=0$ for some $0<q<1$. Let q_1 satisfy $0<q_1<q$. Then $(E, q_1)=(E, q)(E, q_1/q)$ so that $\chi(E, q_1)=\chi(E, q)\cdot\chi(E, q_1/q)=0$. Suppose $q<q_2<1$. Since $\lim q_2^n=0$, choose any n such that $q_2^n<q$. Then $\chi(E, q_2^n)=0$, which implies $\chi(E, q_2)=0$.

Now let $f\in BV[0, 1]$ such that $f(1-0)=f(1)$. Consider the function g defined by $g(t, q)=f(qt)*\varphi_q(t)$, $0<q<1$. Using (1) it is easy to verify that $g(t, q)=f(t)$ for $0\leqq t<q$, and $g(t, q)=f(q)$ for $q\leqq t\leqq 1$. Therefore $\|g(t, q)-f(t)\|\to 0$ as $q\to 1$. Since $\chi(E, q)=0$ for each $0<q<1$, $\chi(H_{g(t,q)})=0$ for each $0<q<1$, and hence $\chi(H_f)=0$.

Any $h\in BV[0, 1]$ can be written in the form $h(t)=f(t)+\lambda\varphi_1(t)$, where $\lambda=\lim\int_0^1 t^n\,df$, and $f(t)=h(t)-\lambda\varphi_1(t)$. Then $\chi(H_h)=\lambda$, so that $\chi=\chi_0$.

Theorem 6. *Let $f\in\mathfrak{A}$, a any constant. Then*

$$\sigma(T_f(M)+a(E, q)) = \overline{\{T_f(z)+aq^{-1+1/z}\colon z\in\bar{D}-\{0\}\}}.$$

Let χ be any multiplicative linear functional on $\mathscr{H}$. Suppose $\chi(M)=z$ for $z\in\bar{D}-\{0\}$. From Theorem 4, $\chi(T_f(M)+a(E, q))=T_f(z)+aq^{-1+1/z}\in\sigma(T_f(M)+a(E, q))$. If $\chi(M)=0$ and $\chi(E, q)=r\neq 0$, then there exists a $z\in\bar{D}-\{0\}$ such that $r=q^{-1+1/z}$. Define $\alpha=\alpha(n)=z\log q/(\log q+2n\,z\pi i)$ for n any integer. Then $q^{-1+1/\alpha}=r$, and $\alpha\to 0$ as $n\to\infty$. Since $f\in\mathfrak{A}$, $T_f(\alpha)\to 0$ as $n\to\infty$. Thus, for all n sufficiently large, $\alpha\in\bar{D}-\{0\}$ and $T_f(\alpha)+aq^{-1+1/\alpha}\in\sigma(T_f(M)+a(E, q))$. The result now follows since the spectrum is closed.

Remark 4. A different proof of Theorem 6, for functions of regular bounded variation, appears in [8].

The following example shows that $\chi(M)=z$, $z=0$ need not imply $\chi(T_f M)=T_f(\chi(M))$. Consider $M+(E, q)\in\mathscr{H}$. From Remark 2, $-1\in\sigma(E, q)$. Define $\{z_n\}$ by $z_n=\log q/(\log q+(2n-1)\pi i)$. Then $z_n\in\bar{D}-\{0\}$ for each n, $z_n\to 0$, and, the multiplicative linear functional on $\mathscr{H}$ defined by $\chi(M)=z_n$ also satisfies $\chi(E, q)=q^{-1+1/z_n}=-1$ for each n. Therefore, by Theorem 6, $z_n-1\in\sigma(M+(E, q))$ for each n. Since the spectrum is closed, $-1\in\sigma(M+(E, q))$. Hence there exists a χ_1 such that $\chi_1(M+(E, q))=-1$. It will now be shown that $\chi_1(M)=0$. Suppose not. Then $\chi_1(M)=z\neq 0$. From Theorem 6, $-1=\chi_1(M+(E, q))=z+q^{-1+1/z}$, or

$q^{-1+1/z} = -(1+z)$ for some $z \in \bar{D} - \{0\}$, which is impossible, since $0<q<1$. Therefore $\chi_1(M)=0$ and $\chi_1(E, q)=-1$.

Thus Theorem 4 cannot be extended to $\bar{D}$. This example also shows that $\chi(M)=0$ does not imply $\chi(E, q)=0$ even though the converse is true from Theorem 5.

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